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On a class of singular Trudinger–Moser type inequalities for unbounded domains in $\mathbb{R}^N$

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ABSTRACT

In this paper, we study a class of Trudinger–Moser inequality associated to the embedding of the Sobolev space $W_0^{1,N}(\Omega)$ into Orlicz spaces for a smooth domain in $\mathbb{R}^N$.

1. Introduction

In this paper, we study the limit case of Sobolev's inequalities; suppose $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space, that is, the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{1,p} = \left\{ \int_\Omega (|\nabla u|^p + |u|^p)\, dx \right\}^{1/p}.$$

It is well known that

$$W_0^{1,p}(\Omega) \subset L^{pN/(N-p)}(\Omega) \quad \text{if } 1 \leq p < N,$$
$$W_0^{1,p}(\Omega) \subset L^\infty(\Omega) \quad \text{if } N < p.$$

The case $p = N$ is the limit case of these embeddings and it is known that

$$W_0^{1,N}(\Omega) \subset L^q(\Omega) \quad \text{for } N \leq q < \infty,$$
$$W_0^{1,p}(\Omega) \not\subset L^\infty(\Omega).$$

When Ω is a bounded domain, we usually use the Dirichlet norm $\|u\|_D = \|\nabla u\|_N$ in the place of $\|u\|_{1,N}$. In this case, we have the famous Trudinger–Moser inequality [1–3] for the limit case $p = N$ which states that

$$\sup_{\|u\|_D \leq 1} \int_\Omega (e^{\alpha |u|^{N/(N-1)}} - 1)\, dx = c_N(\alpha) \cdot |\Omega| \tag{1.1}$$

with

$$c_N(\alpha) \begin{cases} < \infty & \text{if } \alpha \leq \alpha_N \\ = +\infty & \text{if } \alpha > \alpha_N, \end{cases}$$

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where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. The supremum (1.1) becomes infinite for domains Ω with $|\Omega| = +\infty$, and therefore the Trudinger–Moser inequality is not available for unbounded domains. Related inequalities for unbounded domains have been proposed by Adachi and Tanaka [4] and by J.M. do Ó [5]. In [5], it was proved that if $\|\nabla u\|_N \leq 1$, $\|u\|_N \leq M < +\infty$ and $0 < \alpha < \alpha_N$, then there exist $C = C(N, M, \alpha) > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{N/(N-1)})\, dx \leq C(N, M, \alpha), \tag{1.2}$$

where

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

Ruf [6] improved this result when $N = 2$ and Li and Ruf [7] improved this result when $N \geq 3$. They proved that if the Dirichlet norm is replaced by the standard Sobolev norm, then there exists a constant $d > 0$ (independent of Ω) such that for any domain $\Omega \subset \mathbb{R}^N$,

$$\sup_{\|u\|_{1,N} \leq 1} \int_\Omega \Phi_N(\alpha_N |u|^{N/(N-1)})\, dx \leq d. \tag{1.3}$$

Moreover, the inequality is sharp: for any $\alpha > \alpha_N$ the supremum is $+\infty$. On the other hand, Adimurthi and Sandeep [8] extended the Trudinger–Moser inequality with a singular weight; they showed that if Ω is a bounded domain in $\mathbb{R}^N$ containing the origin, $u \in W_0^{1,N}(\Omega)$ and $a \in [0, N)$, then

$$\sup_{\|u\|_D \leq 1} \int_\Omega \frac{(e^{\alpha |u|^{N/(N-1)}} - 1)}{|x|^a}\, dx = C_N(\alpha, a) \cdot |\Omega| \tag{1.4}$$

with

$$C_N(\alpha, a) \begin{cases} < \infty & \text{if } \alpha/\alpha_N + a/N \leq 1 \\ = +\infty & \text{if } \alpha/\alpha_N + a/N > 1. \end{cases}$$

The supremum (1.4) becomes infinite for domains Ω with $|\Omega| = +\infty$, and therefore (1.4) is not available for unbounded domains.

Motivated by Adimurthi and Sandeep [8], Ruf [6] and Li and Ruf [7] the main purpose of this paper is to show the following result.

Theorem 1.1. *If $\Omega \subset \mathbb{R}^N$ is a smooth domain containing the origin with $N \geq 2$, $u \in W_0^{1,N}(\Omega)$ and $K \in L^p(\Omega)$ a nonnegative function such that $K(x) \geq K_0 > 0$ in B_r for some $r > 0$. Then for $\alpha > 0$, $a \in [0, N)$ and $p \in [1, +\infty]$ such that*

$$\frac{\alpha}{\alpha_N} + \frac{a}{N} + \frac{1}{p} \leq 1, \tag{1.5}$$

there exists a constant $\Lambda > 0$ (independent of Ω) such that

$$\sup_{\|u\|_{1,N} \leq 1} \int_\Omega \frac{K(x)\Phi_N(\alpha |u|^{N/(N-1)})}{|x|^a}\, dx \leq \Lambda. \tag{1.6}$$

Moreover, the inequality is sharp in the sense that if $\alpha/\alpha_N + a/N > 1$ the supremum is $+\infty$.

Remark 1.2. Note that when $a = 0$ and $p = +\infty$ our result is identical to the result obtained by Ruf [6] when $N = 2$ and by Li and Ruf [7] when $N \geq 3$.

Remark 1.3. We point out that our results are closely related with some results in [9–12].

Remark 1.4. The Trudinger–Moser result has been extended to Sobolev spaces of higher order and Sobolev spaces over compact manifolds (cf. [9,13]).

Notation. In this work we make use of the following notation.

- $C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at the origin and radius $R > 0$.
- $L^p(\Omega)$ denotes the usual Lebesgue spaces with respect to the norm

$$\|u\|_p = \left(\int_\Omega |u|^p\, dx\right)^{1/p}.$$

- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω, where Ω is a domain of $\mathbb{R}^N$.

The next section will give the proof of Theorem 1.1.

2. Proof of Theorem 1.1

First by Hölder's inequality we have

$$\int_{\Omega} \frac{K(x)\Phi_N(\alpha|u|^{N/(N-1)})}{|x|^a}\,\mathrm{d}x \leq \|K\|_p \left(\int_{\Omega} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x\right)^{1/q},$$

where $1/p + 1/q = 1$. Thus for the first part of the theorem it suffices to study the functional

$$\int_{\Omega} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x.$$

Now, note that

$$\int_{\Omega} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x \leq \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x$$

since any function $u \in W_0^{1,N}(\Omega)$ can be extended by zero outside of Ω, obtaining a function in $W^{1,N}(\mathbb{R}^N)$. Hence, it is sufficient to show that

$$\sup_{\|u\|_{1,N}\leq 1} \int_{\mathbb{R}^N} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x \leq \Lambda. \tag{2.1}$$

Using Schwarz symmetrization, it suffices to show the desired inequality for functions which are nonnegative, radially symmetric and decreasing, $u(x) = u(|x|)$. We divide the integral (2.1) into two parts, with $\rho_0 > 0$ to be chosen

$$\int_{\mathbb{R}^N} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x = \int_{B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}}\,\mathrm{d}x. \tag{2.2}$$

To estimate the first integral in (2.2), let $v(\rho) = u(\rho) - u(\rho_0)$ if $0 \leq \rho \leq \rho_0$ and $v(\rho) = 0$ if $\rho \geq \rho_0$. Notice that $v \in W_0^{1,N}(B_{\rho_0})$. Using the fact that the function $g : (0, +\infty) \to \mathbb{R}$ given by

$$g(t) = \frac{(t+1)^{N/(N-1)} - t^{N/(N-1)} - 1}{t^{1/(N-1)}}$$

is bounded, we have a positive constant $A = A(N)$ such that

$$u^{N/(N-1)}(\rho) \leq v^{N/(N-1)}(\rho) + Av^{1/(N-1)}(\rho)u(\rho_0) + u^{N/(N-1)}(\rho_0).$$

On the other hand, by the Young inequality, we have

$$v^{1/(N-1)}u(\rho_0) \leq \frac{v^{N/(N-1)}(\rho)u^N(\rho_0)}{N} + \frac{N-1}{N} \leq \frac{v^{N/(N-1)}(\rho)u^N(\rho_0)}{N} + 1. \tag{2.3}$$

By the "Radial Lemma" (cf. [14]), we have

$$u(\rho_0) \leq \frac{1}{\rho_0}\left(\frac{N}{\omega_{N-1}}\right)^{1/N} \|u\|_N.$$

Therefore, we have

$$u^N(\rho_0) \leq \frac{N}{\omega_{N-1}\rho_0^N}\|u\|_N^N. \tag{2.4}$$

The combination of (2.3) and (2.4), implies that

$$v^{1/(N-1)}(\rho)u(\rho_0) \leq v^{N/(N-1)}(\rho)\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N} + 1. \tag{2.5}$$

Hence

$$u^{N/(N-1)}(\rho) \leq v^{N/(N-1)}(\rho)\left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right) + A + u^{N/(N-1)}(\rho_0).$$

Using again the "Radial Lemma", we have

$$u^{N/(N-1)}(\rho) \leq v^{N/(N-1)}(\rho)\left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right) + A + \frac{1}{\rho_0^{N/(N-1)}}\left(\frac{N}{\omega_{N-1}}\right)^{1/(N-1)} \|u\|_N^{N/(N-1)}.$$

As $\|u\|_N \le 1$, we get

$$u^{N/(N-1)}(\rho) \le v^{N/(N-1)}(\rho)\left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right) + A + \frac{1}{\rho_0^{N/(N-1)}}\left(\frac{N}{\omega_{N-1}}\right)^{1/(N-1)}$$
$$= v^{N/(N-1)}(\rho)\left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right) + d(\rho_0).$$

Hence

$$u(\rho) \le v(\rho)\left[1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right]^{(N-1)/N} + d(\rho_0)^{(N-1)/N} =: w(\rho) + d(\rho_0)^{(N-1)/N}.$$

By assumption

$$\int_{B_{\rho_0}} |\nabla v|^N \, dx = \int_{B_{\rho_0}} |\nabla u|^N \, dx \le 1 - \|u\|_N^N,$$

and hence

$$\int_{B_{\rho_0}} |\nabla w|^N \, dx = \left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right)^{N-1} \int_{B_{\rho_0}} |\nabla v|^N \, dx$$
$$\le \left(1 + A\frac{\|u\|_N^N}{\omega_{N-1}\rho_0^N}\right)^{N-1} (1 - \|u\|_N^N).$$

Since

$$\left(1 + \frac{A}{\omega_{N-1}\rho_0^N} s^N\right)^{N-1} (1 - s^N) \le 1 \quad \text{if } s \in [0, 1] \text{ and } \frac{A}{\omega_{N-1}\rho_0^N} \le 1/(N-1),$$

we obtain

$$\int_{B_{\rho_0}} |\nabla w|^N \, dx \le 1.$$

However, $u(\rho)^{(N-1)/N} \le w(\rho)^{N/(N-1)} + d(\rho_0)$, then

$$\int_{B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}} \, dx \le \int_{B_{\rho_0}} \frac{e^{\alpha q|u|^{N/(N-1)}}}{|x|^{aq}} \, dx$$
$$\le e^{\alpha q d(\rho_0)} \int_{B_{\rho_0}} \frac{e^{\alpha q|w|^{N/(N-1)}}}{|x|^{aq}} \, dx.$$

But $w \in W_0^{1,N}(B_{\rho_0})$, $\int_{B_{\rho_0}} |\nabla w|^N dx \le 1$ and by (1.5) we have $aq < N$ and $\alpha q/\alpha_N + aq/N \le 1$, thus follows by (1.4) that there exist $C_1 > 0$ such that

$$\int_{B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}} \, dx \le e^{\alpha q d(\rho_0)} C_1. \tag{2.6}$$

For the second integral, choosing ρ_0 sufficiently large such that $\rho_0 \ge 1$, we have

$$\int_{\mathbb{R}^N \setminus B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}} \, dx \le \int_{\mathbb{R}^N \setminus B_{\rho_0}} \Phi_N(\alpha q|u|^{N/(N-1)}) \, dx.$$

Hence, by (1.3) there exist $C_2 > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_{\rho_0}} \frac{\Phi_N(\alpha q|u|^{N/(N-1)})}{|x|^{aq}} \, dx \le C_2. \tag{2.7}$$

Then (2.6) and (2.7) follow the first part of the theorem. Next we will show that (1.6) does not hold if $\alpha/\alpha_N + a/N > 1$. Consider the Moser function (cf. [1]):

$$M_n(x) = (\omega_{N-1})^{-1/N} \begin{cases} (\log n)^{(N-1)/N} & \text{if} \quad |x| \le r/n \\ \dfrac{\log\left(\frac{r}{|x|}\right)}{(\log n)^{1/N}} & \text{if} \quad r/n \le |x| \le 1 \\ 0 & \text{if} \quad |x| \ge r. \end{cases}$$

Hence $M_n(\cdot) \in W^{1,N}(\mathbb{R}^N)$, $supp(M_n) = \overline{B}_r$, $\|\nabla M_n\|_N = 1$ and $M_n \rightharpoonup 0$ in $W^{1,N}(\mathbb{R}^N)$. Then by the compact embedding, we may assume that $\int_{B_r} M_n^p \, dx \to 0$ for $p \geq 1$, then $\|M_n\|_{1,N} \to 1$, for n large. Given any fixed α such that $\alpha/\alpha_N + a/N > 1$, we take $\beta \in (\alpha_N(N-a)/N, \alpha)$, then

$$\alpha(M_n/\|M_n\|_{1,N})^{N/(N-1)} > \beta M_n^{N/(N-1)}$$

when n is sufficiently large. So, we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \frac{K(x)\Phi_N(\alpha|M_n/\|M_n\|_{1,N}|^{N/(N-1)})}{|x|^a} \, dx \geq K_0 \lim_{n\to+\infty} \int_{B_r} \frac{(e^{\beta M_n^{N/(N-1)}} - 1)}{|x|^a} \, dx = +\infty.$$

This completes the proof of the result.

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